Aerodynamic Penetration and Radius as Unifying Concepts in Flight Mechanics

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The scale of atmospheric entry processes and of certain other problems in vehicle dynamics may be quantified in terms of a vehicle parameter that has the dimension of length. Two such lengths, the "aerodynamic penetration" $2m/\rho AC_D$ and the "aerodynamic radius" $2m/\rho AC_L$, have simple physical interpretations and tend to be constant multiples of the characteristic dimensions of geometrically similar vehicles of constant density. The aerodynamic penetration may be used as a linear scale factor to generalize trajectory studies if aerodynamic drag is important, and the aerodynamic radius may be used if aerodynamic lift is important. Several problems involving the deceleration and flight mechanics of entry vehicles, airplanes, and ballistic projectiles are discussed to show the application of these parameters to well-known situations.

Nomenclature

A = vehicle reference area for drag coeff
C_D = vehicle drag coeff $\equiv (2D/\rho v^2 A)$
C_{D_p} = parasite drag coeff (lifting airframe)
C_L = vehicle lift coeff $\equiv (2L/\rho v^2 A)$
D = drag, or dimensionless derivative operator $\equiv \tau(d/dt)$
$\mathfrak{D}$ = load factor in drag direction $\equiv D/mg$
g = acceleration of gravity
h = altitude
h_e = atmospheric scale height ($\rho = \rho_0 e^{-h/h_e}$)
Δh = altitude above potential energy altitude for equilibrium-level flight
H = altitude in scale heights $\equiv h/h_e$
ΔH = altitude in scale heights above reference altitude
L = lift
m = vehicle mass
N = equilibrium normal load factor
r = dimensionless speed ratio $\equiv v/v_r$
R = flight path or equilibrium orbit radius
r_p = speed ratio at peak of trajectory (ballistics)
s = distance along path
s_D = aerodynamic penetration $\equiv (2m/\rho AC_D)$
s_{D_p} = aerodynamic penetration based on C_{D_p}
s_{D_0} = aerodynamic penetration based on ρ_0
s_L = aerodynamic radius $\equiv (2m/\rho AC_L)$
s_{L_0} = aerodynamic radius based on ρ_0
t = time
t_D = deceleration time const $\equiv s_D/v_r$
$\bar{u}$ = independent variable in Chapman's equation
v = speed
v_r = fixed reference speed defined as required
X = horizontal trajectory coordinate
Y = vertical trajectory coordinate
Z = dependent variable in Chapman's equation
γ = flight path angle with respect to the horizontal
δ = perturbation quantity (prefix)
ϵ = disturbance parameter for phugoid equation
ζ = oscillation damping ratio
λ = time derivative operator $\equiv d/dt$
Λ = oscillation wavelength
μ = coeff of rolling friction (wheeled vehicle)
ρ = air density
ρ_0 = air density at planet surface (isothermal atmosphere)
τ_D = ballistic time const $\equiv (s_D/g)^{1/2}$
τ_L = phugoid time const $\equiv (s_L/g)^{1/2}$
ω = oscillation frequency (rad/unit time)
ω_n = undamped natural frequency

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Introduction

A NEED exists to generalize the results of studies in atmospheric flight mechanics. It is the purpose of this paper to satisfy this need, for many applications, by showing the usefulness of two vehicle parameters as linear scale factors in trajectory studies.

The first parameter is the "aerodynamic penetration." It arises in connection with processes dominated by aerodynamic drag. For an object decelerated by aerodynamic drag alone, the equation of motion is

$$dv/dt = v(dv/ds) = -(\rho AC_D/2m)v^2 \tag{1}$$

which may be written as

$$dv/v = -(\rho AC_D/2m)ds = -(1/s_D)ds \tag{1a}$$

In Eq. (1a) s_D is defined as the aerodynamic penetration.

For the case of constant aerodynamic penetration, the solution of Eq. (1a) is

$$v/v_r = r = e^{-(s-s_r)/s_D} \tag{2}$$

where r is a speed ratio, defined as the ratio of the variable speed to a reference speed v_r existing at a reference distance s_r. The aerodynamic penetration is the distance that an object of mass m travels in air of density ρ for aerodynamic drag proportional to the square of the speed to reduce its speed by a factor of e.

The second parameter is the "aerodynamic radius." It arises in connection with processes dominated by aerodynamic lift. For an object accelerated normal to its direction of flight by aerodynamic lift alone, the equation of motion is

$$v(d\gamma/dt) = v(v/R) = (\rho AC_L/2m)v^2 \tag{3}$$

where γ is the upward path angle and R is the radius of flight path curvature. This equation can be written in the form

$$1/R = (\rho AC_L/2m) = 1/s_L \tag{3a}$$

from which it follows that s_L, the aerodynamic radius, is the radius of flight path curvature caused by aerodynamic lift alone. For geometrically similar objects of constant density, both the aerodynamic penetration and radius are proportional to a characteristic linear dimension of the object if ρ, C_D, and C_L are constant.

The terms s_D or s_L can be considered to be a certain number of calibers or wing spans for a certain class of vehicle, and so convey fairly accurate notions of the scale of the trajectory of a particular vehicle for which general solutions exist in terms of the parameter s_D or s_L, as the case may be. Table 1 gives values of these parameters for some familiar vehicles.

The remainder of the paper is devoted to the derivation of general solutions of the type just described. All the solutions presented are for the two following equations of motion:

$$dv/dt = -D/m - g \sin\gamma \quad (4)$$

$$v(d\gamma/dt) = L/m - g(\cos\gamma)(1 - v^2/gR) \quad (5)$$

Two general classes of solution are discussed: those involving rectilinear motion [Eq. (4)] and those involving curvelinear motion [Eqs. (4) and (5)]. Some of the solutions are well known in terms of different parameters.

Application (Solutions Involving Rectilinear Motion)

a. Deceleration of a Bullet

The simplest case of pure aerodynamic deceleration along a straight trajectory has been discussed already in the Introduction. The process described by Eqs. (1, 1a, and 2) is appropriate to the deceleration of a bullet in a ballistic range, where the trajectory is essentially straight and horizontal. If the speed can be measured at several points along the range, a plot of the logarithm of the speed against distance should be a straight line, and the slope of the line should be a measure of the inverse aerodynamic penetration. Departures from linearity are an indication of nonconstant drag coefficient.

The speed can be found also as a function of time. Equation (1) can be written in the form

$$dv/dt = v_r(dr/dt) = -(\rho A C_D/2m)v_r^2 r^2 \quad (1b)$$

with the solution

$$\frac{(1 - r)}{r} = \frac{(t - t_r)}{(s_D/v_r)} \quad (6)$$

where $r = v/v_r$ and $r = 1$ when $t = t_r$.

b. Deceleration of an Automobile

An automobile may be allowed to coast to a stop on a level road against the action of aerodynamic drag, assumed proportional to the square of the speed, and of rolling friction, assumed independent of speed. The equation of motion is

$$dv/dt = -(\rho A C_D/2m)v^2 - \mu g \quad (7)$$

where μ is the coefficient of rolling friction. Equation (7) can be written in the form

$$dr/(r^2 + 1) = -(v_r/s_D)dt \quad (8)$$

where v_r, the reference speed, is chosen to normalize the integral of the left-hand side of Eq. (8). The resulting reference speed is that speed that makes the aerodynamic drag equal to the rolling friction

$$\tfrac{1}{2}\rho A C_D v_r^2 = \mu m g \quad (9)$$

The solution of Eq. (8) is given by Eq. (10), where the penetration time s_D/v_r is the time required to travel the penetration distance at the reference speed. Thus,

$$\frac{(t - t_r)}{(s_D/v_r)} = \frac{\pi}{4} - \arctan r \quad (10)$$

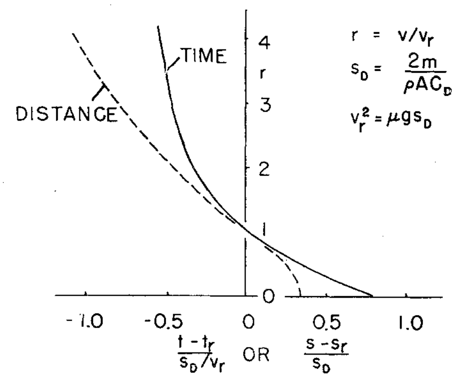

Fig. 1 Deceleration of an automobile.

An alternative solution for Eq. (8) is given by Eq. (11), where t_{stop} is the time at which the car stops rolling. Thus,

$$\frac{(t_{stop} - t)}{(s_D/v_r)} = \arctan r \quad (11)$$

Equation (7) can be written also in terms of a distance variable

$$v(dv/ds) = v_r^2 r(dr/ds) = -(v_r^2 r^2/s_D) - g \quad (12)$$

with the solution

$$(s - s_r)/s_D = \tfrac{1}{2}\log[2/(r^2 + 1)] \quad (13)$$

where v_r is defined as before [Eq. (9)]. The solutions given by Eqs. (10) and (13) are plotted in Fig. 1. In Ref. 1, the author suggests this process as an experimental technique for determining the aerodynamic drag and rolling friction of an actual automobile. This publication continues to give coast-down time histories for each automobile tested. Careful analysis of these data would require consideration of the increase of apparent mass because of angular momentum of the wheels and drive line, as well as corrections to account for rolling friction variations with speed.

Table 1 A few representative lengths

Vehicle	Size	Aerodynamic penetration	Aerodynamic radius
Rifle bullet	Caliber = 7.62 mm	1325 m	. . .
Peugeot 404 sedan	Wheelbase = 2.65 m	2350 m	. . .
HBV "Diamant" sailplane	Wingspan = 15.0 m	4650 m[a]	59.6 m[b]
Mercury capsule	Heatshield diam = 1.89 m	825 m[c]	. . .
X-20 "Dynasoar"	Wingspan = 6.1 m	340 m[c,d]	340 m[c,d]

[a] Based on C_{D_p}
[b] Based on C_L for $(L/D)_{max}$.
[c] Based on $\rho_0 = 1.39$ kg/m³.
[d] Based on C_L, for $(L/D) = 1$.
For comparison: $h_e = 7$ km; $R = (20{,}000/\pi) = 6370$ km.

c. Deceleration of a Sailplane in Level Flight

Measurement of the steady-state gliding performance of sailplanes is very difficult because of the lightly damped phugoid motion described later. Moeller[2] suggested that the drag be determined by constraining the sailplane to level flight and finding v (and hence dv/dt) as functions of time. The equations to be satisfied are (1) and

$$C_D = C_{D_p} + [dC_D/d(C_L^2)](C_L^2) \quad (14)$$

$$\tfrac{1}{2}\rho v^2 A C_L = mg \quad (15)$$

Cone[3] has noted that Eq. (14) may not be valid at the beginning of the deceleration process, where the sailplane presumably is close to the starting vortex associated with the abrupt pull-up from a high-speed dive to level flight, and again, near the end of the process, where the gradient of

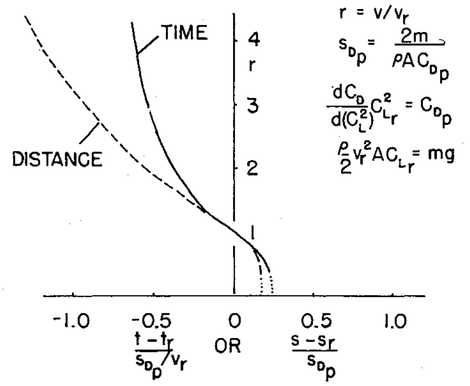

Fig. 2 Deceleration of a sailplane.

circulation along the trailing vortices, times the wing span, is appreciable compared to the bound circulation.

Nevertheless, relatively simple analytic solutions of the simultaneous Eqs. (1, 14, and 15) exist for both distance and time variables. The distance solution is given by

$$(s - s_r)/s_{D_p} = (\tfrac{1}{4}) \log[2/(r^4 + 1)] \quad (16)$$

where

$$s_{D_p} = 2m/\rho AC_{D_p} \quad (17)$$

and v_r is the speed for best L/D, given by

$$C_{L_r} = \{C_{D_p}/[dC_D/d(C_L^2)]\}^{1/2} \quad (18)$$

$$\tfrac{1}{2}\rho v_r^2 AC_{L_r} = mg \quad (19)$$

The time solution is more complicated. It is

$$\frac{(t - t_r)}{t_{D_p}} = \frac{(2)^{1/2}}{4}\left\{\frac{1}{2}\log\left[\frac{r^2 + (2)^{1/2}r + 1}{r^2 - (2)^{1/2}r + 1}\cdot\frac{2 - (2)^{1/2}}{2 + (2)^{1/2}}\right] + \frac{\pi}{2} - \arctan\left(\frac{(2)^{1/2}r}{1 + r^2}\right)\right\} \quad (20)$$

where

$$t_{D_p} = s_{D_p}/v_r \quad (21)$$

The solutions given by Eqs. (16) and (20) are plotted in Fig. 2. These relations were derived by the author in 1959 as a Christmas present for the late A. Raspet.

d. Deceleration of a Ballistic Missile

Allen and Eggers[4] were among the first to discuss the deceleration of an object entering an isothermal atmosphere at a steep angle and at speeds large compared with the terminal velocity. The equation of motion is

$$dv/dt = v(dv/dh)(dh/ds) = -(\rho_0 AC_D/2m)e^{-(h/h_e)}(v^2) \quad (22)$$

where the density in the atmosphere is an exponential function of the altitude h

$$\rho = \rho_0 e^{-(h/h_e)} \quad (23)$$

From kinematics,

$$dh/ds = \sin\gamma \quad (24)$$

If the flight path remains straight during the entry process, Eq. (22) can be written in the form

$$dv/v = -(h_e/s_{D_0}\sin\gamma)e^{-(h/h_e)}\,d(h/h_e) \quad (25)$$

In Eq. (25), s_{D_0} is the aerodynamic penetration at the (projected) bottom of the isothermal atmosphere. The well-known solution to Eq. (25) is

$$r = v/v_r = e^{-(h_e/s_{D_0}\sin\gamma)e^{-(h/h_e)}} = e^{-Ke^{-H}} \quad (26)$$

where v_r is the "entrance" velocity outside the atmosphere (at large values of h), and γ is (just this once) positive downwards.

The deceleration is given by

$$dv/dt = -(v_r^2 \sin\gamma/h_e)(rdr/dH)$$
$$= -(v_r^2 \sin\gamma/h_e)Ke^{-(H+2Ke^{-H})} \quad (27)$$

Examination of Eq. (27) shows that the peak deceleration always occurs when $1 + 2Ke^{-H} = 0$, or when $r = e^{-1/2} = 0.606$. This corresponds to the dimensionless altitude

$$\frac{h_{(dv/kt)\max}}{h_e} = \log 2K = \log\left(\frac{2h_e}{s_{D_0}\sin\gamma}\right) \quad (28)$$

The general altitude may be specified as in incremental altitude above that producing maximum deceleration,

$$(h - h_{(dv/dt)\max})/h_e = \Delta H \quad (29)$$

Particularly simple expressions for the speed and deceleration can be written in terms of this incremental altitude,

$$r = e^{-(1/2e)^{-\Delta H}} \quad (30)$$

$$\frac{(dv/dt)}{(dv/dt)_{\max}} = e^{(1-\Delta H - e^{-\Delta H})} \quad (31)$$

$$(dv/dt)_{\max} = -(\tfrac{1}{2}e)(v_r^2 \sin\gamma/h_e) \quad (32)$$

Figure 3 is a plot of the speed-altitude relation given by Eq. (30) and the deceleration-altitude relation given by Eq. (31). Note that the peak deceleration given by Eq. (32) depends entirely upon the entrance conditions v_r and $\sin\gamma$; the aerodynamic penetration s_{D_0} serves only to regulate the altitude at which peak deceleration is experienced [Eq. (28)]. Larger values of aerodynamic penetration corresponds to deeper penetrations of the atmosphere (i.e., lower altitudes) before peak deceleration is experienced.

Application (Solutions Involving Curvilinear Motion)

a. Classical Exterior Ballistics

The equations of motion for a ballistic projectile moving in an atmosphere of constant density and in a uniform gravitational field are

$$dv/dt = -(\rho AC_D/2m)v^2 - g\sin\gamma \quad (33)$$

$$v(d\gamma/dt) = -g\cos\gamma \quad (34)$$

The grouping $(\rho AC_D/2m)$ is the reciprocal of the aerodynamic penetration and is assumed constant. The speed is made dimensionless with a reference speed corresponding to the

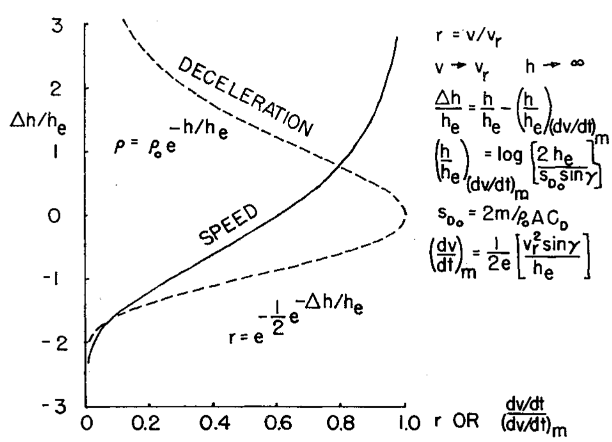

Fig. 3 Deceleration of a ballistic missile.[4]

terminal velocity in free fall,

$$\tfrac{1}{2}\rho v_r^2 A C_D = mg \tag{35a}$$

$$(\rho A C_D/2m)v_r^2 = g \tag{35b}$$

$$v_r^2 = g s_D \tag{35c}$$

$$r^2 = (v/v_r)^2 = v^2/g s_D \tag{36}$$

Introducing these parameters permits Eqs. (33) and (34) to be written in the form

$$(s_D/g)^{1/2}(dr/dt) = -r^2 - \sin\gamma \tag{33a}$$

$$(s_D/g)^{1/2}(d\gamma/dt) = -\cos\gamma/r \tag{34a}$$

A dimensionless time is defined together with a dimensionless derivative operator,

$$t' = \frac{t}{\tau_D} = \frac{t}{(s_D/g)^{1/2}} \tag{37}$$

$$D = \frac{d}{d(t/\tau_D)} = \tau_D\left(\frac{d}{dt}\right) = \left(\frac{s_D}{g}\right)^{1/2}\left(\frac{d}{dt}\right) \tag{38}$$

leading to the following dimensionless equations of motion:

$$Dr = -r^2 - \sin\gamma \tag{33b}$$

$$D\gamma = -(\cos\gamma)/r \tag{34b}$$

The horizontal and vertical coordinates of the trajectory are given by the relations

$$X = \int v \cos\gamma\, dt = v_r\tau_D \int r \cos\gamma d(t/\tau_D)$$

$$= (g s_D)^{1/2}(s_D/g)^{1/2}\int r \cos\gamma dt'$$

$$X = s_D \int r \cos\gamma dt' \tag{39}$$

$$Y = \int v \sin\gamma dt = s_D \int r \sin\gamma dt' \tag{40}$$

Equations (33b) and (34b) are integrated to find r and γ as functions of t'; these results are substituted in Eqs. (39) and (40). It can be seen that the coordinates of the trajectory are proportional to the aerodynamic penetration for different projectiles fired at a fixed muzzle angle γ_0 and a muzzle velocity r_0, that is, a fixed fraction of the terminal velocity v_r.

Alternatively, it is possible to eliminate the time variable from Eqs. (33) and (34) to obtain

$$dr/d\gamma - r \tan\gamma = r^3/\cos\gamma \tag{41}$$

with the solution

$$(1/r_p^2) - (1/r \cos\gamma)^2 = \tan\gamma \sec\gamma + \log \tan(\gamma/2 + \pi/4) \tag{42}$$

According to Becker and Cranz,[5] this solution was given by J. Bernoulli in 1719. In Eq. (42), r_p is the fraction of terminal

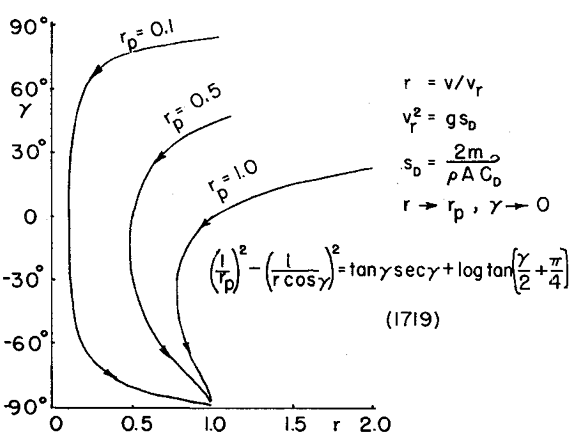

Fig. 4 Bernoulli's hodograph equation.

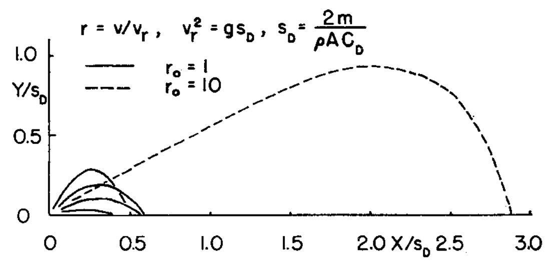

Fig. 5 Ballistic trajectories.

velocity at the peak of the trajectory, arising as a constant of integration from the solution of Eq. (41). Figure 4 is a plot of the hodograph relation implied by Eq. (42).

The coordinates of the trajectory also can be determined by integrating with respect to the path angle. Thus, if Eq. (42) is solved for $r = F(\gamma)$, then

$$X = \int v \cos\gamma\, dt = \int v \cos\gamma(-v\, d\gamma/g \cos\gamma) = \int (v^2/g)d\gamma$$

$$X = s_D \int r^2 d\gamma = s_D \int [F(\gamma)]^2 d\gamma$$

Similarly,

$$Y = \int v \sin\gamma dt = s_D \int [F(\gamma)]^2 \tan\gamma\, d\gamma$$

The coordinates of a practical trajectory for a projectile with a known value of s_D are found by scaling general solutions of the type plotted in Fig. 5 by an amount proportional to s_D.

b. Lanchester's Phugoid

The equations of motion for an airplane flying at constant angle of attack, with powerplant thrust equal to airframe drag at all airspeeds, in air of constant density and in a uniform gravitational field, are

$$dv/dt = -g \sin\gamma \tag{43}$$

$$v(d\gamma/dt) = (\rho A C_L/2m)v^2 - g \cos\gamma \tag{44}$$

The grouping $(\rho A C_L/2m)$ is the reciprocal of the aerodynamic radius and is assumed constant. The speed is made dimensionless with a reference speed corresponding to the speed for equilibrium level flight. Thus,

$$\tfrac{1}{2}\rho v_r^2 A C_L = mg \tag{45a}$$

$$(\rho A C_L/2m)v_r^2 = g \tag{45b}$$

$$v_r^2 = g s_L \tag{45c}$$

$$r^2 = (v/v_r)^2 = v^2/g s_L \tag{46}$$

Introducing these parameters permits Eqs. (43) and (44) to be written in the form

$$(s_L/g)^{1/2}(dr/dt) = -\sin\gamma \tag{43a}$$

$$(s_L/g)^{1/2}(d\gamma/dt) = r - (\cos\gamma)/r \tag{44b}$$

Again, if $(s_L/g)^{1/2}$ is recognized as a time constant τ_L, the following expressions can be written for the horizontal and vertical coordinates of the trajectory,

$$X = \int v \cos\gamma dt = v_r\tau_L \int \cos\gamma\, d(t/\tau_L)$$

$$= (g s_L)^{1/2}(s_L/g)^{1/2}\int r \cos\gamma d(t/\tau_L)$$

$$X = s_L \int r \cos\gamma d(t/\tau_L) \tag{47}$$

Similarly

$$Y = \int v \sin\gamma dt = s_L \int r \sin\gamma d(t/\tau_L) \tag{48}$$

The coordinates of a given trajectory characterized by a certain solution of $r = f_r(t/\tau_L)$ and $\gamma = f_\gamma(t/\tau_L)$ can be scaled to a certain vehicle having a particular value of aerodynamic

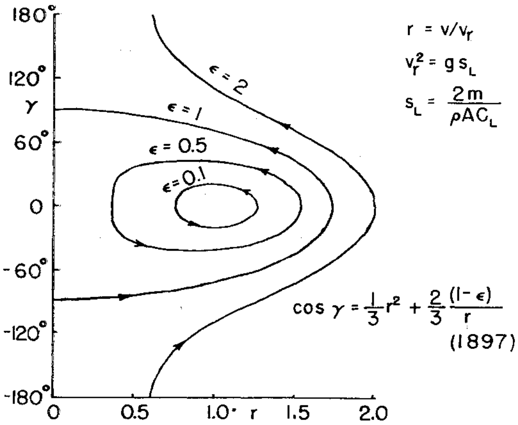

Fig. 6 Lanchester's phugoid equation.

radius s_L. F. W. Lanchester showed[6] that t can be eliminated in Eqs. (43) and (44), to yield

$$\frac{d(-\cos\gamma)}{d(\Delta h)} - \frac{(-\cos\gamma)}{2(s_L/2 - \Delta h)} = \frac{1}{s_L} \tag{49}$$

where

$$2g\Delta h = v_r^2 - v^2$$

or

$$v^2 = 2g(s_L/2 - \Delta h) \tag{50}$$

Equation (49) has the solution

$$\cos\gamma = (\tfrac{1}{3})(1 - 2\Delta h/s_L) + (\tfrac{2}{3})(1 - \epsilon)/(1 - 2\Delta h/s_L)^{1/2} \tag{51}$$

or

$$\cos\gamma = (\tfrac{1}{3})r^2 + (\tfrac{2}{3})(1 - \epsilon)/r \tag{51a}$$

where ϵ is a disturbance coefficient arising as a constant of integration. Increasingly positive values of ϵ correspond to increasing departures from equilibrium level flight. Figure 6 shows the hodograph relation between the speed and path angle implicit in Eq. (51a). Lanchester was careful to point out that he discovered this result in 1897. Figure 7 shows the phugoid trajectories appropriate to different values of the disturbance coefficient ϵ. Small values of ϵ give rise to an undulating motion with a wavelength approaching $[\pi(2)^{1/2}]$ s_L for vanishing ϵ; $\epsilon = 1$ corresponds to a trajectory made up of a series of whip stalls connected by perfect circular arcs of radius $(\tfrac{2}{3})s_L$; values of ϵ greater than 1 correspond to steady looping flight. Finally, as ϵ becomes very large, the trajectory approaches a stationary circular loop of radius s_L. Lanchester felt that the phugoid motion was the essence of flight mechanics, and wished to call it the "flying" motion; hence, the Greek name "phugoid", which he supposed, to imply flight. Unfortunately, it implies flight only in the sense of escape, as in "fugitive."

c. Linearized Phugoid

A difficulty with Lanchester's analysis is its neglect of dissipative effects that cause damping of actual phugoid motion. If the motion is a small perturbation superimposed on equilibrium-level flight at the reference speed v_r, the speed ratio is

$$r = 1 + \delta r \tag{52}$$

and the path angle is

$$\gamma = 0 + \delta\gamma \tag{53}$$

If the lift is proportional to the square of the speed,

$$L/mg = (\rho A C_L/2m)(v^2/g) \tag{54}$$

the perturbation of lift is given by

$$\frac{\delta L}{mg} = v_r \left[\frac{\partial\,(L/mg)}{\partial v}\right]_{v=v_r} \times \left(\frac{\delta v}{v_r}\right) = 2\delta r \tag{55}$$

The perturbation drag is given similarly by

$$\frac{\delta D}{mg} = \left[\frac{2}{L/D}\right]\delta r \tag{56}$$

where (L/D) is the equilibrium lift/drag ratio. If the powerplant thrust is proportional to v raised to the k power,

$$\delta T/mg = \left[\frac{k}{(L/D)}\right]\delta r \tag{57}$$

The perturbations of weight components along and perpendicular to the flight path are

$$\delta(mg\,\sin\gamma)/mg = \delta\gamma \tag{58}$$

$$\delta(mg\,\cos\gamma)/mg = 0 \tag{59}$$

Substituting these perturbation forces in the equations of motion and rearranging yield

$$\left[\frac{v_r}{g}\frac{d}{dt} + \frac{(2-k)}{(L/D)}\right]\delta r + \delta\gamma = 0 \tag{60}$$

$$-2\,\delta r + (v_r/g)(d/dt)\delta\gamma = 0 \tag{61}$$

This pair of linear equations with constant coefficients has the characteristic equation ($\lambda = d/dt$),

$$\left(\frac{v_r}{g}\right)^2\left\{\lambda^2 + \left[\frac{(2-k)}{(L/D)}\right]\left(\frac{g}{v_r}\right)\lambda + 2\left(\frac{g}{v_r}\right)^2\right\} = 0 \tag{62}$$

which corresponds to a damped oscillation of frequency

$$\omega = \omega_n(1 - \zeta^2)^{1/2} = (g/v_r)(2)^{1/2}(1 - \zeta^2)^{1/2} \tag{63}$$

and damping ratio

$$\zeta = \left[\frac{(2)^{1/2}}{(4)}\right]\left[\frac{(2-k)}{(L/D)}\right] \tag{64}$$

The wavelength of the oscillation is

$$\Lambda = 2\pi v_r/\omega = \pi(2)^{1/2}(v_r^2/g)(1 - \zeta^2)^{-1/2} \tag{65}$$

which checks the Lanchester results for undamped small amplitude motions.

Application of Mueller's "time vector concept"[7] facilitates determination of the mode shape appropriate to the phugoid oscillation. The disturbed airplane describes a decaying elliptic orbit about its equilibrium position, as shown in Fig. 8. In the case of a high-performance sailplane,

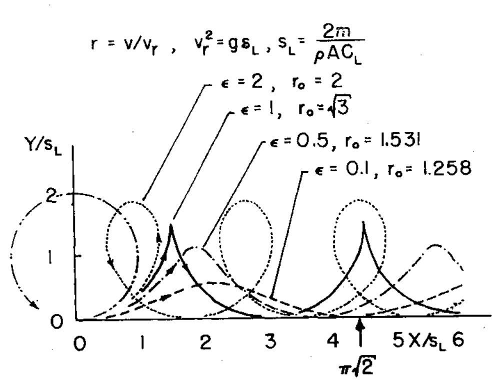

Fig. 7 Phugoid trajectories.

Eq. (64) shows that the damping of the phugoid oscillation is very small, and that the perturbation components of the rate of sink, which may be appreciable fractions of the steady rate of sink (which it is desirable to measure), come and go with a period of 30 sec or more.

It may be shown that the amplitude of the vertical component of the phugoid perturbation orbit is

$$|\delta h| = (v_r^2/g)|\delta r| = s_L|\delta r| \quad (66)$$

The "spring" force in the linearized phugoid oscillation just described is the change of lift with airspeed given by Eq. (55). Equation (66) shows that for high-speed aircraft the vertical height perturbations associated with a given level of airspeed perturbation may be appreciable fractions of the atmospheric scale height h_e. It becomes necessary to take account of the additional spring forces caused by the change of lift with air density whenever the change of height, times the density gradient, becomes comparable to the gradient of dynamic pressure with airspeed, times the change of airspeed. This corresponds to equilibrium airspeeds of

$$\tfrac{1}{2}v_r^2 = gh_e \quad (67)$$

or more. The same condition is satisfied by Eq. (68), as follows:

$$\tfrac{1}{2}s_L = h_e \quad (68)$$

At still higher equilibrium speeds, it becomes necessary to account for the reduction of equilibrium lift in level flight by reason of approach to satellite circular orbit speed, the air density gradient with height, and the gravitational acceleration gradient with height. Equations (69–71) are linear equations of motion with constant coefficients that include these effects:

$$\left[\left(\frac{v_r}{g_0}\right)\left(\frac{d}{dt}\right) + N\frac{(2-k)}{(L/D)}\right]\delta r + \delta\gamma + (0)\,\frac{\delta h}{h_e} = 0 \quad (69)$$

$$-2\delta r + (v_r/g_0)(d/dt)\delta\gamma + [N - (1+N)(h_e/R_0)]\delta h/h_e = 0 \quad (70)$$

$$(O)\delta r - (v_r/h_e)\delta\gamma + (d/dt)\delta h/h_e = 0 \quad (71)$$

In this array, which applies to airplanes with air-breathing engines, R_0 is the equilibrium orbit radius, g_0 is the acceleration of gravity at the radius R_0, and N is the equilibrium normal load factor,

$$L_0/mg_0 = 1 - (v_r^2/g_0R_0) = N$$

The characteristic equation for the system represented by Eqs. (69–71) is third order. Its factors are a small real root, corresponding to a weak height stability, and a complex quadratic corresponding to the phugoid oscillation. The frequency of the oscillation approaches the following limits

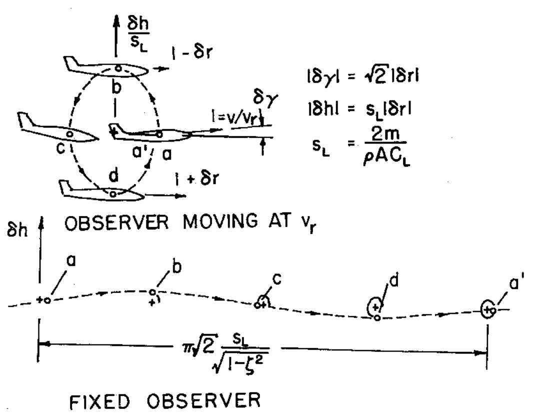

Fig. 8 Linearized phugoid.

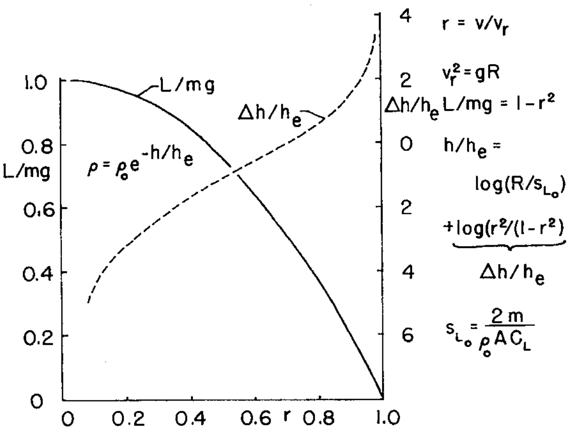

Fig. 9 Equilibrium gliding entry.[9]

for different values of $v_r = (g_0s_L)^{1/2}(1 - N)^{1/2}$:

$\omega_n = (2)^{1/2}(g_0/v_r)$ for $v_r \ll (g_0h_e)^{1/2}$ and $v_r \ll (g_0R_0)^{1/2}$

$\omega_n = (2)^{1/2}(1 + \tfrac{1}{2}v_r^2/g_0h_e)^{1/2}(g_0/v_r)$

for $v_r \sim (g_0h_e)^{1/2}$ and $\ll (g_0R_0)^{1/2}$

ω_n approaches g_0/v_r as v_r approaches $(g_0R_0)^{1/2}$. Etkin gives a similar treatment in Ref. 8.

d. Equilibrium Gliding Entry

Consider a glider constrained to fly at constant angle of attack (and therefore constant C_L and L/D) in an isothermal atmosphere. The flight path remains nearly level, so that the vehicle loses speed according to the equation

$$dv/dt = -D/m = -(D/L)(L/m) \quad (72)$$

The "equilibrium" condition consists of the assumption of nearly level flight at constant orbit radius R,

$$v(d\gamma/dt) = 0 = L/m - g\cos\gamma(1 - v^2/gR) \cong L/m - g(1 - v^2/gR) \quad (73)$$

If we take the speed ratio as

$$r^2 = v^2/gR \quad (74)$$

and introduce the range angle ϕ

$$v = R(d\phi/dt) \quad \text{or} \quad dt = R(d\phi/dv) \quad (75)$$

then Eq. (72) may be written in the form

$$dv/dt = (gR)(rdr/Rd\phi) = -(D/L)g(1 - r^2) \quad (76)$$

Equation (76) is integrated to yield

$$\phi_2 - \phi_1 = \tfrac{1}{2}(L/D)\log[(1 - r_2^2)/(1 - r_1^2)] \quad (77)$$

Equation (73) may be written in the form

$$L/m = gR(\rho_0AC_L/2m)e^{-h/h_e}r^2 = g(1 - r^2) \quad (78)$$

Alternatively, with $s_{L_0} = (2m/\rho_0AC_L)$

$$e^{-h/h_e} = (s_{L_0}/R)(1 - r^2)/r^2 \quad (79)$$

The speed-altitude relation implicit in Eq. (79) can be written in the form

$$h/h_e = (h/h_e)_{\text{design}} + \Delta h/h_e = \log(R/s_{L_0}) + \log[(r^2/(1 - r^2)] \quad (80)$$

From this, it follows that vehicle design s_{L_0} serves to establish only the particular altitude at which a given fraction of circular orbit speed is achieved. The variation of altitude $\Delta h/h_e$ with fraction of circular orbit speed is independent of vehicle design. Figure 9 shows the variation of normal load

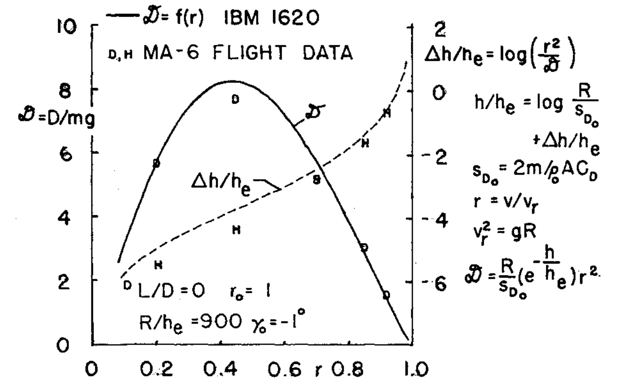

Fig. 10 Nonequilibrium entry.

factor $L/mg = (1 - r^2)$ and incremental altitude $\Delta h/h_e = \log[(r^2/(1 - r^2)]$ with fraction of circular orbit speed. Saenger gives a similar analysis in Ref. 9. The solution for a particular vehicle is known as soon as s_{L_0} is calculated, in much the same way that the Allen-Eggers solution for ballistic entry is known for a particular vehicle once s_{D_0} is calculated.

e. Nonequilibrium Entry

Consider a vehicle entering at this isothermal atmosphere surrounding a spherical planet. Neglecting the speed of the atmosphere caused by the rotation of the planet in comparison to the entry speed, and neglecting the component of vehicle weight along the trajectory compared to aerodynamic drag, the equations of motion are

$$dv/dt = -D/m \tag{81}$$

$$v(d\gamma/dt) = (L/D)(D/m) - g\cos\gamma(1 - v^2/gR) \tag{82}$$

The entry is assumed to take place at constant angle of attack so that the drag is given by

$$D/m = (\rho_0 AC_D/2m)e^{-h/h_e}v^2 \tag{83}$$

Upon introduction of a dimensionless speed variable $v/v_r = r$ where $v_r^2 = gR$, Eq. (81) can be written in the form

$$(gR)^{1/2}(dr/dt) = -g(R/s_{D_0})(e^{-H})r^2 \tag{81a}$$

where $H = h/h_e$. H. Ashley suggested introduction of a dimensionless variable

$$\mathfrak{D} = D/mg = (R/s_{D_0})(e^{-H})r^2 \tag{84}$$

which corresponds physically to the longitudinal load factor. Equation (81a) can be written in the form

$$dr/dt = -(g/R)^{1/2}\mathfrak{D} \tag{81b}$$

Equation (82) can be written in the form

$$v(d\gamma/dt) = (L/D)(D/m) - g\cos\gamma(1 - r^2) \tag{82a}$$

or, alternatively,

$$d\gamma/dt = (gR)^{-1/2}(1/r)(L/D)(g\mathfrak{D}) - g\cos\gamma(1 - r^2)/(gR)^{1/2}r \tag{82b}$$

Dividing Eq. (82b) by (81b) yields

$$d\gamma/dr = -(L/D)(1/r) + \cos\gamma(1 - r^2)/\mathfrak{D}r \tag{85}$$

From kinematics,

$$dh/dt = v\sin\gamma = (dh/dr)(dr/dt) = (gR)^{1/2}r\sin\gamma \tag{86}$$

Solving for $\sin\gamma$,

$$\sin\gamma = (gR)^{-1/2}(h_e/R)(dH/dr)(dr/dt) \tag{87}$$

The relation for dH/dr needed in Eq. (87) is found by differentiating Eq. (84),

$$d\mathfrak{D}/dr = (2/r)\mathfrak{D} - (dH/dr)\mathfrak{D} \tag{88}$$

Or, solving for dH/dr,

$$dH/dr = (2/r) - (1/\mathfrak{D})(d\mathfrak{D}/dr) \tag{89}$$

Substituting Eq. (89) in Eq. (87),

$$\sin\gamma = -(h_e/R)[2\mathfrak{D}/r^2 - (1/r)(d\mathfrak{D}/dr)] \tag{90}$$

Differentiating Eq. (90)

$$d(\sin\gamma)/dr = \cos\gamma(d\gamma/dr) = -(h_e/R)(1/r)[(d^2\mathfrak{D}/dr^2) - (3/r)(d\mathfrak{D}/dr) + (4/r^2)\mathfrak{D}] \tag{91}$$

Setting the expressions for $d\gamma/dr$ from Eqs. (85) and (91) equal to each other and rearranging the result yield

$$d^2\mathfrak{D}/dr^2 - (3/r)d\mathfrak{D}/dr + (4/r^2)\mathfrak{D} = (R/h_e)[\cos^2\gamma(1 - r^2)/\mathfrak{D} - \cos\gamma(L/D)] \tag{92}$$

Equation (92) is related to the equation derived by Dean Chapman in Ref. 10. Chapman's equation differs from this one in that it is written for polar coordinates with the origin at the center of the planet. The speed variable $\bar{u}$ in Chapman's equation is the ratio of the horizontal component of the speed to circular orbit speed. Chapman's Z function has no simple physical significance, but it is related to the $\mathfrak{D}$ function by the following equation

$$(R/h_e)^{1/2}\bar{u}Z = \mathfrak{D}\cos\gamma \tag{93}$$

Equation (92) can be solved numerically to yield deceleration speed histories for general families of vehicles having arbitrary initial entrance conditions. Once $\mathfrak{D}$ has been found as a function of r for a restricted family of vehicles, the altitude speed history for a particular vehicle can be found from the relation

$$h/h_e = (h/h_e)_{\text{design}} + (\Delta h/h_e) = \log(R/s_{D0}) + \log(r^2/\mathfrak{D}) \tag{94}$$

which follows from the definition of $\mathfrak{D}$ in Eq. (89). Figure 10 compares a solution of Eq. (92) for the case of shallow nonlifting entry with flight data from the entry of the Mercury MA-6 capsule. The agreement is reasonably good.

Figure 11 compares a solution of Eq. (92) for the case of shallow lifting entry with $L/D = 1$ and $\gamma_{\text{initial}} = -1°$ at $r_{\text{initial}} = 1$ with the equilibrium glide solution given by Eqs. (80) and (76). The nonequilibrium entry solution is seen to oscillate about the equilibrium glide solution. It can be

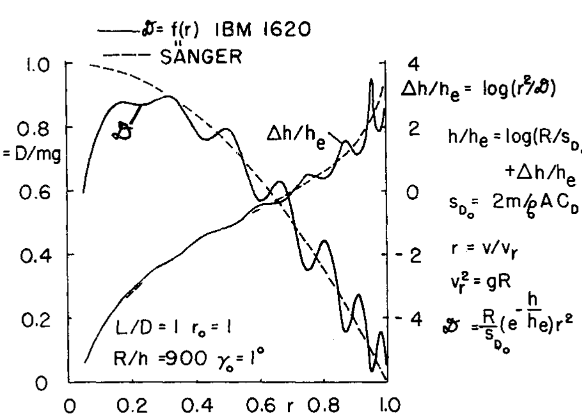

Fig. 11 Nonequilibrium entry.

shown that the period and damping of these oscillations may be approximated by a linearized phugoid analysis that accounts for the atmospheric density gradient and the equilibrium lift appropriate to the particular value of r at which the oscillation occurs.

Conclusion

Two classes of problems in flight mechanics have been analyzed from a uniform viewpoint in such a way as to show the occurence of two vehicle design parameters, the aerodynamic penetration, $s_D = 2m/\rho AC_D$ and the aerodynamic radius $s_L = 2m/\rho AC_L$. These parameters, which have the dimension of length, are proportional to the linear dimensions of geometrically similar vehicles of constant density. The trajectories appropriate to the situations in flight mechanics discussed are found to have coordinates, especially altitude coordinates, which are simply related to either the aerodynamic penetration or the aerodynamic radius. For these situations, and presumably for others for which general solutions can be found, a particular solution applicable to a particular vehicle may be found by simply calculating the particular value of the aerodynamic penetration or radius, as the case may be, together with the appropriate reference speed.

References

[1] Larrabee, E. E., "Measuring car drag," Road Track **12,** 24–28 (February 1961).

[2] Moeller, C. A., "New performance test method," Soaring **25,** 3 (October 1961).

[3] Cone, C. D., "The induced drag of a decelerating airfoil," Aero Revue (Swiss) **90,** 267–270, 334–336 (May–June 1965).

[4] Allen, H. J. and Eggers, A. J., Jr., "A study of the motion and aerodynamic heating of missiles entering the earth's atmosphere at high supersonic speeds," NACA TN 4047 (October 1947); also NACA RM A53D28 (October 1947).

[5] Cranz, C. and Becker, K., *Lehrbuch der Ballistik: Vol. I, Exterior Ballistics*, transl. by His Majesty's Stationary Office (1921).

[6] Lanchester, F. W., *Aerodynamics* (Arnold Constable, London, 1907).

[7] Mueller, R. K., "The graphical solution of stability problems," J. Aeronaut. Sci. **4,** 324–337 (1937).

[8] Etkin, B., "Longitudinal dynamics of a lifting vehicle in orbital flight," J. Aerospace Sci. **28,** 825–826 (1961).

[9] Bredt, I. and Saenger, E., "A rocket drive for long range bombers," Deutsche Luftfahrtforschung UM 3538 (1944); U. S. Navy Bureau of Aeronautics CGD-32.

[10] Chapman, D., "An approximate analytical method for studying entry into planetary atmospheres," NACA TN 4276 (May 1958); also NASA TR R-11 (1959).